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Numerical Integration of the Vlasov Equation*

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This work describes a numerical method for the solution of the nonlinear Vlasov equation. The distribution function is expanded in a series of orthogonal polynomials, namely the Chebyshev polynomials. The conditions under which this expansion is valid are discussed. Recurrence effects are eliminated by formally adding a damping term to the eigenvalues of the truncated system. Nonlinear effects have been simulated by an amount of information corresponding to less than 500 "particles."

I. INTRODUCTION

The use of transform methods for the numerical solution of the nonlinear Vlasov equation has been discussed and reviewed in several publications [1, 2]. One has to deal with the one-dimensional Vlasov equation

$$\partial f(x, v, t)/\partial t + v(\partial f/\partial x) - E(x, t)(\partial f/\partial v) = 0. \tag{1}$$

Supplemented with the Poisson equation

$$\frac{\partial E}{\partial x} = \left(1 - \int f\, dv\right). \tag{2}$$

The units used in Eqs. (1) and (2) are the inverse plasma frequency $\omega_p^{-1} = (4\pi n_0 e^2/m)^{-1/2}$, the thermal velocity $v_T = (KT/m)^{1/2}$, and the Debye length $\lambda_D = (KT/4\pi n_0 e^2)^{1/2}$.

The expansion of the distribution function in velocity space in terms of orthogonal polynomials, namely the Hermite polynomials, has been particularly studied in [3], and more recently by Knorr [4], who simulated linear effects by an amount of information corresponding to less than 100 "particles," nonlinear effects by less than 500 "particles." This has been effected by applying a method prescribed in [2] to eliminate the recurrence effects, namely the addition of a real damping term to the eigenvalues of the truncated system.

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The expansion of the velocity dependence of the distribution function in terms of Hermite polynomials has been essentially dictated by the fact that no other classical polynomials possess such a simple expression for their derivatives, although they converge slowly to the correct solution. The question thus arises on the possibility of using other classical orthogonal polynomials possessing a more rapid convergence. Since in numerical simulation one is compelled to truncate the velocity dependence of the distribution function at, say, $v = \pm V_m$ (thus neglecting the very few particles with $v > V_m$ for V_m sufficiently large), the use of plynomials orthogonal over a finite interval may present an attractive alternative. With the aid of a given finite number of these polynomials one might represent the distribution function with a higher accuracy in the velocity range of interest.

It has been found that the Chebyshev polynomials are appropriate for this purpose. The choice of these polynomials and the derivation of the Chebyshev representation of the Vlasov equation will be presented in the next section, and the last section will present our results and conclusion.

II. The Chebyshev Representation

A. *The Expansion of the Distribution Function with Polynomials Orthogonal Over Finite Interval*

We rewrite Eq. (1) in the form

$$\frac{\partial f(x, \tilde{v}, t)}{\partial t} + \tilde{v} V_m \frac{\partial f(x, \tilde{v}, t)}{\partial x} - \frac{E(x, t)}{V_m} \frac{\partial f(x, \tilde{v}, t)}{\partial \tilde{v}} = 0 \tag{3}$$

where $\tilde{v}$ varies in the closed interval $(-1, 1)$ and V_m is the maximum value of the velocity. The distribution function $f(x, \tilde{v}, t)$ is expanded in the following series

$$f(x, \tilde{v}, t) = \sum_{\nu=0}^{\infty} b_\nu(x, t)\, U_\nu(\tilde{v})\, w(\tilde{v}). \tag{4}$$

The polynomials U_ν are to be determined; they obey the orthogonality relation

$$\int_{-1}^{+1} U_\nu(\tilde{v})\, U_\mu(\tilde{v})\, w(\tilde{v})\, d\tilde{v} = \alpha_\nu \delta_{\nu\mu} \tag{5}$$

where the weight function $w(\tilde{v})$ is also to be determined.

We choose periodic boundary conditions and substitute from Eq. (4) into Eq. (3). Integrating over phase-space, using Eq. (5) we get

$$\frac{1}{L} \alpha_0 \frac{\partial}{\partial t} \int_{-L/2}^{L/2} b_0(x, t)\, dx - \frac{1}{L} \int_{-L/2}^{L/2} \frac{E}{V_m} [f(x, \tilde{v}, t)]_{\tilde{v}=-1}^{\tilde{v}=+1}\, dx = 0 \tag{6}$$

From Eq. (4) we have

$$[f(x, \tilde{v}, t)]_{\tilde{v}=-1}^{\tilde{v}=+1} = \sum_{\nu=0}^{\infty} b_\nu [U_\nu(1)\, w(1) - U_\nu(-1)\, w(-1)]. \tag{7}$$

It results that Eq. (6) can be accurately verified if and only if the sum in Eq. (7) is extended over a sufficiently large number of polynomials. This contradicts the basic aim of our present work, which is the economical simulation of the Vlasov equation by decreasing as much as possible the number of polynomials used. This contradiction can be overcome if we choose polynomials whose weight function w vanishes at the boundary of the interval, i.e., $w(\pm 1) = 0$. (The Jacobi, Gegenbauer, or Chebyshev polynomials are examples.) Among these polynomials, the Chebyshev polynomials of the second kind, U_ν, have a simple recursion relation [5]

$$\tilde{v} U_\nu(\tilde{v}) = (1/2)[U_{\nu+1}(\tilde{v}) + U_{\nu-1}(\tilde{v})]. \tag{8}$$

In this case the weight function is $w = (1 - \tilde{v}^2)^{1/2}$ and $\alpha_\nu = \pi/2$.

The expansion in Eq. (4) is now written

$$f(x, \tilde{v}, t) = \sum_{\nu=0}^{\infty} b_\nu(x, t)\, U_\nu(\tilde{v})(1 - \tilde{v}^2)^{1/2}. \tag{9}$$

This expansion is imposing on $f(x, \tilde{v}, t)$ a nonphysical boundary condition, namely that $f(x, \pm 1, t) = 0$ for all time t, which is not fulfilled by the system in Eq. (1) and (2), since the acceleration term of the Vlasov equation can accelerate particles for arbitrary high velocities. However, one can, for the sake of mathematical and computational convenience, introduce some modification on the system of Eq. (1) and (2), without changing fundamentally the physics involved in the problem. In order to keep the expansion in Eq. (9) valid, one can use an acceleration term in Eq. (3) which vanishes at the boundary; i.e., we rewrite Eq. (3) in the form

$$\frac{\partial f}{\partial t} + V_m \tilde{v} \frac{\partial f}{\partial x} - \frac{E}{V_m} \frac{\partial}{\partial \tilde{v}} [(1 - \tilde{v}^{2q}) f] = 0. \tag{10}$$

The effect of this modification is clearly to keep the distribution f equal to zero at the boundary $\tilde{v} = \pm 1$, for all time t, thus legitimizing the expansion in Eq. (9). As it can be seen from Fig. 1, the higher the value of q, the closer the value $(1 - v^{2q})$ to 1 for the region where the distribution function is appreciably different from zero, and hence the closer is Eq. (10) to the Vlasov equation given in Eq. (3).

We now ask what conservation laws does the newly derived system verify. Integrating Eq. (10) over phase space, we can easily verify that the number of particles is conserved. Momentum and energy are not conserved by Eq. (10); however, as we previously mentioned, the higher the value of q, the closer is the

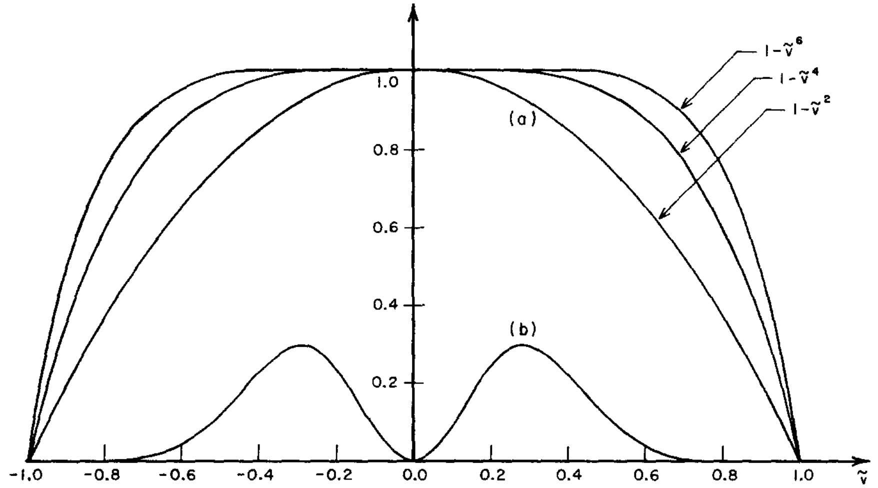

FIG. 1. (a) Plot of the curves $(1 - \tilde{v}^{2q})$ against $\tilde{v}$ for $q = 1, 2, 3$. (b) Plot of the distribution function $f = (1/(2\pi)^{1/2})\, \tilde{v}^2 V_m{}^2\, e^{-1/2\, \tilde{v}^2 V_m{}^2}$ against $\tilde{v}$ for $V_m = 5.0$.

quantity $(1 - \tilde{v}^{2q})$ to 1 in the region where the distribution function is appreciably different from zero (see Fig. 1) and hence, the closer the moments of Eq. (10) to the moments of the Vlasov equation.

B. *The Truncation of the Infinite Matrix*

We have represented the v-dependence of the distribution function by a series of Chebyshev polynomials in Eq. (9). We set $E = 0$ for the moment and substitute from Eq. (9) in Eq. (10), we get

$$\frac{\partial b_\nu}{\partial t} + \frac{V_m}{2} \frac{\partial}{\partial x} (b_{\nu+1} + b_{\nu-1}) = 0 \tag{11}$$

When the infinite system in Eq. (11) is truncated by setting $b_N(x, t) = 0$, the continuous eigenvalue spectrum of this system is replaced by the set of discrete finite eigenvalues of the truncated system. This can be easily seen by setting

$$b_\nu(x, t) = b_\nu \exp(ikx + \Lambda t) \tag{12}$$

in Eq. (11). We get

$$i\Lambda 2b_\nu / kV_m = b_{\nu+1} + b_{\nu-1}\,. \tag{13}$$

This is the recursion relation for the Chebyshev polynomials. Thus for b_ν we have the solution

$$b_\nu = U_\nu(i\Lambda / kV_m). \tag{14}$$

The system is finite if $b_N = 0$, i.e.,

$$U_N(i\Lambda/kV_m) = 0 \qquad \text{or} \qquad \Lambda_\alpha = -(ikV_m)\, Z_\alpha^N \qquad \alpha = 1, 2,..., N \tag{15}$$

where Z_α^N is the αth root of the Chebyshev polynomial U_N. Substitute by the previous results in Eq. (9), we get

$$f = \sum_{\nu=0}^{N} \sum_{\alpha=0}^{N} a_\alpha U_\nu(Z_\alpha^N)\, U_\nu(\tilde{v})(1 - \tilde{v}^2)^{1/2} \exp(ikx - ikV_m Z_\alpha^N t) \tag{16}$$

where the constants a_α are to be determined from the initial value of f. Clearly, the results in Eq. (16), calculated for a truncated system, represent an almost periodic function in time.

C. *The Addition of Damping*

In order to make the solution of the truncated system more similar to the true solution, a remedy was suggested in [2] which consists of adding a real part to the eigenvalues calculated in Eq. (15). This corresponds physically to the presence of dissipative terms on the right-hand side of Eq. (11), such as a "collision term," for example. The study of such term for Hermite polynomials expansion has been given by Knorr and Shoucri [6]. In order to get the same effects using the Chebyshev polynomials, we follow the same steps as in [6] and look for a "collision" operator C such that

$$C(U_\nu(1 - \tilde{v}^2)^{1/2}) = a_\nu U_\nu(1 - \tilde{v}^2)^{1/2}. \tag{17}$$

It is straightforward to verify that for the operator C defined by

$$C \equiv (1 - \tilde{v}^2)(\partial^2/\partial\tilde{v}^2) - \tilde{v}(\partial/\partial\tilde{v}) + 1 \tag{18}$$

one has

$$C(U_\nu(1 - \tilde{v}^2)^{1/2}) = -\nu(\nu + 2)\, U_\nu(1 - \tilde{v}^2)^{1/2}. \tag{19}$$

More generally, one has

$$C^{2r+1}(U_\nu(1 - \tilde{v}^2)^{1/2}) = -[\nu(\nu + 2)]^{2r+1}\, U_\nu(1 - \tilde{v}^2)^{1/2}. \tag{20}$$

Accordingly, Eq. (10) is modified to

$$\frac{\partial f}{\partial t} + \tilde{v} V_m \frac{\partial f}{\partial x} - \frac{E}{V_m} \frac{\partial}{\partial \tilde{v}} [(1 - \tilde{v}^{2q}) f] = \eta \frac{\partial^2}{\partial x^2} C^{2r+1}(f). \tag{21}$$

η is a constant. The operator $\partial^2/\partial x^2$ helps make the Fourier modes having the highest k value (and hence, the lowest recurrence time, since the recurrence time $T \sim N/k$) be damped selectively.

We make use of Eq. (21) to damp the filamentation of the Vlasov equation.

Having determined the "collision" operator C as given in Eq. (18), it is interesting to determine the "equilibrium" distribution towards which C drives the distribution function, i.e., we are looking for the distribution f_0 such that

$$C(f_0) = 0. \tag{22}$$

From Eq. (20), it results immediately that for $\nu = 0$, we have $U_0 = 1$ and

$$C((1 - \tilde{v}^2)^{1/2}) = 0, \tag{23}$$

hence,

$$f_0 = (1 - \tilde{v}^2)^{1/2}. \tag{24}$$

III. The Finite Difference Scheme

The system in Eq. (11) is hyperbolic and it is more advantageous in this case to use for the numerical calculation a leapfrog scheme, as pointed out by Knorr [4]. The stability of the leapfrog scheme is investigated following the same steps as in [4] and leads to the stability condition:

$$\Delta t/\Delta x < |\lambda_N|^{-1} \tag{25}$$

where λ_N is the largest root of the Chebyshev polynomial U_N, which is always less than one.

The difference scheme for the dissipation term on the right-hand side of Eq. (21) has been calculated using a Dufort–Frankel scheme, to increase the stability of the system.

The two level leapfrog scheme has been initialized using a two-level Lax–Wendroff scheme, which accomplished an initialization of second order in Δt. The Lax–Wendroff scheme has been also used to initialize the leapfrog scheme continuously after a number of time steps, to prevent the two levels of the leapfrog scheme from drifting apart, in the same way it has been discussed by Knorr [4]. In the present results, the Lax–Wendroff initialization was repeated every 40 time steps.

IV. Results and Conclusions

The example of the symmetric two-stream instability has been studied with the initial condition

$$f(x, v, 0) = (1/(2\pi)^{1/2})\, v^2 \exp(-(1/2)\, v^2)(1 + A \cos kx)$$

with $A = 0.1$ and $k = 1/2$. (We have subtracted from $f(x, v, 0)$ the quantity

$$f_M = (1/(2\pi)^{1/2})\, V_m{}^2 \exp(-(1/2)\, V_m{}^2)(1 + A \cos kx)$$

in order to have a distribution which is zero at $v = V_m$). A typical solution is given in Figs. 2a and b, which has been calculated by setting $q = 3$ in Eq. (21). Figure 2a gives the magnitude of the first two modes as a function of time. In Fig. 2b the total electric energy is plotted linearly in time. It follows the characteristic exponential growth, saturation, and oscillations of the electric field due to the trapping of the particles. These plots have the physical features of those reported in Figs. 4a and b of Ref. [4], and have been obtained with the equivalent of 480 particles. The damping reported on the right-hand side of Eq. (21) has been applied only to the last 13 coefficients, out of a matrix of 30 (for the other terms, the damping was too small and could be neglected). The curves in Figs. 2a and b correspond to $\eta = 5$ and $r = 2$.

At this point, a comparison between the respective accuracies of the Chebyshev polynomials and the Hermite polynomials representation is necessary. One can define the accuracy of a polynomial representation as being the maximum distance between two consecutive zeroes of the polynomial, which characterizes the "resolution" one can get with this representation. For the Hermite polynomials

$$H_{en_H}(v) \sim \cos((2n_H)^{1/2}\, v),$$

hence, the distance between two consecutive zeros

$$\Delta v_H = \pi/n_H^{1/2}.$$

The mth zero of the Chebyshev polynomial U_{n_c} is given by [5]

$$X_m^{(n_c)} \sim \cos m/(n_c + 1)\pi.$$

In our case, the distance between two consecutive zeroes of a Chebyshev polynomial is given by

$$\Delta v_c = \pi V_m/n_c + 1.$$

An equal accuracy of the two representations is attained if

$$\Delta v_H = \Delta v_c\,;$$

hence,

$$\pi V_m/(n_c + 1) = \pi/n_H^{1/2}.$$

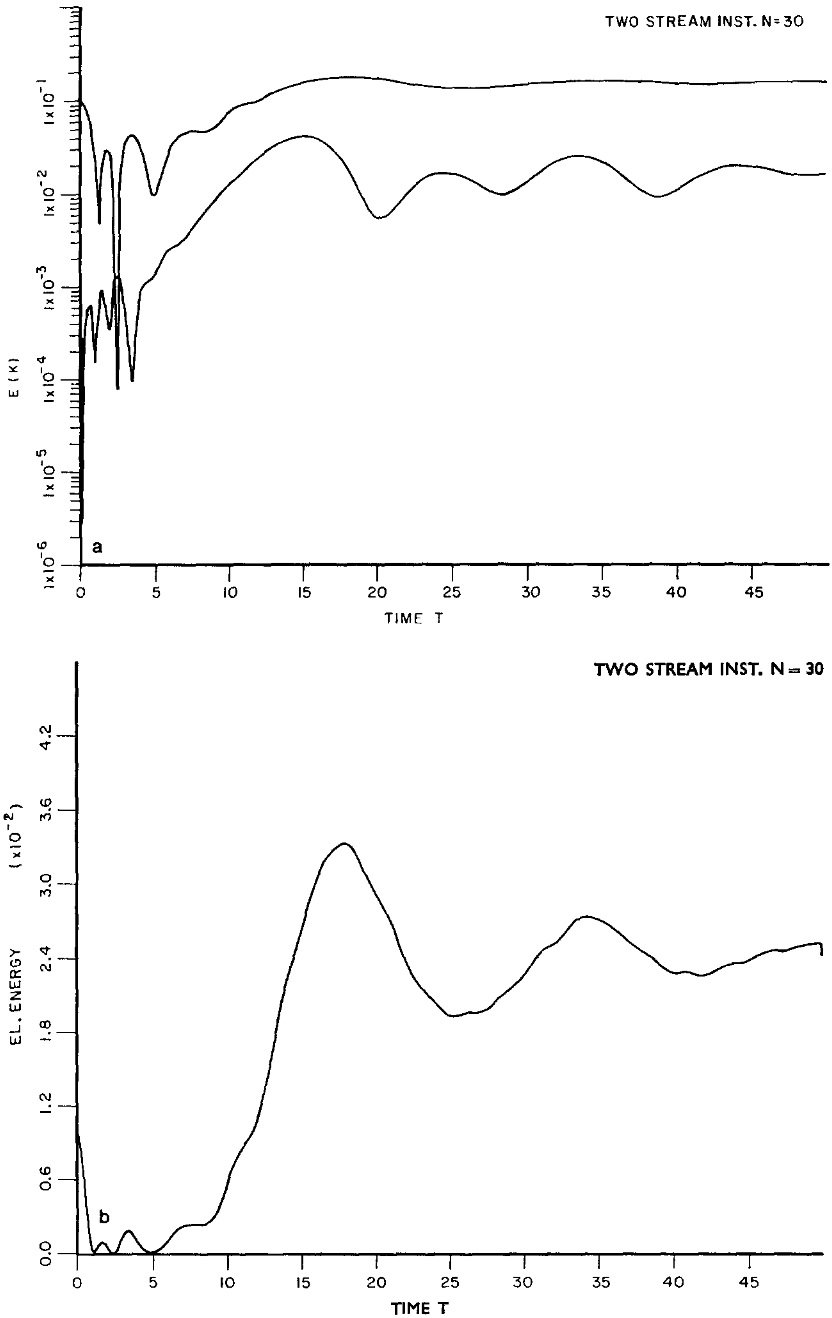

FIG. 2. Plot of the total electric field energy for a two-stream instability with the initial condition $f(x, v, 0) = (1/(2\pi)^{1/2})\, v^2 e^{-1/2v^2} (1 + A \cos kx)$ with $A = 0.1$ and $k = 1/2$. These results are calculated for $q = 3$, $\eta = 5.0$ and $r = 2$.

The point of equal computational effort (i.e., requiring the same number of polynomials in both representation) is given by $n_c = n_H = n$, hence,

$$n \approx V_m^2,$$

i.e., if $V_m = 5$ (which is the case in our present calculation), then it is more advantageous to take Hermite polynomials if less than 25 polynomials are to be used. If more than 25 polynomials are to be used, then it is more advantageous to use Chebyshev polynomials (the results presented in this paper use 30 Chebyshev polynomials).

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